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THE INVERSE GALOIS PROBLEM AND MAEDA'S ASSUMPTION

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ABSTRACT

According to Maeda's hypothesis on level 1 Eigen forms, the simple group $PSL_2\mathbb{F}(p^d)$ exists as the Galois group of a number field ramifying solely at p for every positive even d and every p in density one set of primes.

Keywords-*Inverse Galois Problem, Maeda's Assumption, Eigen forms*

Introduction

This paper's objective is to support the use of automorphic forms to solve the inverse Galois problem for certain finite Lie type groups. $PSL_2\ \mathbb{F}\ (\ell^d)$, groups have had some encouraging results recently, such as [2, 11], and more general groups have seen some encouraging results as well [1, 6, 7]. The fundamental concept is to analyse the pictures of the residual Galois representations connected to various automorphic forms over $\mathbb{Q}$. One can only get infinity or positive-density outcomes for now. The lack of control over the fields of coefficients of the associated automorphic forms appears to be the key technical impediment to upgrading the given results to density 1.

If we adhere to the simplest situation, i.e. automorphic forms for GL_2 over $\mathbb{Q}$, we can use Maeda's hypothesis on the coefficient fields of level 1modular forms to prove our assumption. The control over the coefficient fields offered by Maeda's hypothesis suffices to provide the following strong result on the inverse Galois problem, which we use to highlight the modular approach's promise.

.**Theorem 1.1** ***Assume the following form of*** **Maeda's conjecture** ***on level 1 modular forms:***

"The coefficient field $\mathbb{Q}f := \mathbb{Q}\ (a_n(f) \mid n \in N)$ has degree $d_k := \dim_{\mathbb{C}} S_k(1)$ and the Galois group of its normal closure over $\mathbb{Q}$ is the symmetric group $Sd_k$for any k and any normalized Eigen formf $\in S_k(1)$ (the space of cuspidal modular forms of weight k and level 1)."

(a) "*Let* $\mathbf{2 \leq d \in N}$*be even. In that case, the number field K/$\mathbb{Q}$ with Galois group isomorphic to $PSL_2\mathbb{F}(p^d)$ has density 1, and the set of prime psuch that it is ramified only at p has density 1.*"

(b) "*Let* **1** ≤ ***d*** ∈ℕ*be odd. In that case, the number field K/*ℚ *with Galois group isomorphic to* $PSL_2\mathbb{F}(p^d)$ *has density 1, and the set of prime p such that it is ramified only at p has density 1.*"

Hypothesis 1.2 in [5] was Maeda's assumption. This product has passed safety checks up to a weight of pounds (see [4]). We also highlight Tsaknias's proposed [9] to extend a weaker version of Maeda's assumption to square-free levels. Natural density or Dirichlet density can be used interchangeably throughout the paper. If Theorem 1.1 is beneficial, then it can certainly be reformulated. Assume that the weights up to *B* in Maeda's hypothesis have been checked. The density of the sets in the theorem can be lowered explicitly for all "***d*** ≤ **dim**ℂ***SB***" (1) depending on *B*. The remainder of the paper contains the proof for Theorem 1.1. For this, Ribet [8], Chebotarev's density theorem, a few combinatorics in symmetric groups, and Galois theory are used as foundations.

2 Proof

This section proves the thesis statement's key point. We follow the standard of considering the symmetric group "S_n"to be the group of permutations of the set "{1, 2, .. . , n}".

2.1. Derivatives of the symmetric Galois group divide into primes.

"It is well known that the splitting behaviour of unramified primes in a simple extension *K*(*a*)/*K* may be read off from the cycle type of the Frobenius, considered as an element of the permutation group of the roots of the minimum polynomial a. In this section, we present a somewhat non-standard proof. In [10], the reduction of the minimum polynomial of *a* is factorised into irreducibles. This is a more standard proof.

If *M/K* is a separable field extension of degree *n*, then *L/M* is the Galois closure of *M* over *K*. There exists a *M* such that *M* = *K*(*a*). according to the theorem of the primitive element . This is the minimum polynomial of *a* over *K* , and these is *a*'s roots in $a = a_1, a_2, . . . , a_n$.$\psi$: ***G*** := **Gal(*L/K*)** → ***Sn*,**is an injective group homomorphism that maps ***H*** **:= Gal(*L/M*)** onto with the permutation given by $S_n(1) \cap \psi(G)$."

Proposition 2.1

"Assume the preceding set-up with K a number field. Let 𝔭*be aprime of K and* 𝔓*a prime of L dividing* 𝔭*. We suppose that***𝔓/𝔭***is unramified. Then* 𝔭*the cycle lengths in the cycle decomposition of* $\psi(\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}) \in Sn$ *are precisely theresidue degrees of the primes of M lying above*𝔭.𝔓*"*

Proof

"Let ***g*** ∈**Gal (*L/K*)**. Denote by Let ***g*** ∈**Gal (*L/K*).** Denote by $\mathbf{Frob}_{\mathfrak{P}/\mathfrak{p}}$ the Frobenius element of $g\,\mathfrak{P}/\mathfrak{p}$ in Gal(*L/K*) and **by** $f(g\mathfrak{P}\cap M)/\mathfrak{p}$ the inertial degree of the prime $g\,\mathfrak{P}\cap M$ of *M* over 𝔭.Write ϕ := FrobP/p for short. the Frobenius element of $g\,\mathfrak{P}/\mathfrak{p}$ inGal(*L/K*) and by $f(gP\cap M)/p$ the inertial degree of the prime $g\,\mathfrak{P}\cap M$ of *M* over p.Write $\varphi := \mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}$short". We have

$$f_{(g\mathfrak{P}\cap M)/\mathfrak{p}} = \min_{i\in\mathbb{N}}(\mathrm{Frob}^{i}_{g\mathfrak{P}/\mathfrak{p}} \in H) = \min_{i\in\mathbb{N}}(\varphi^{i} \in g^{-1}Hg).$$

From this we obtain the equivalences:

$$\begin{aligned} & \exists g \in G: \ f_{(g\mathfrak{P}\cap M)/\mathfrak{p}} = d, \\ \Leftrightarrow \ & \exists g \in G: \ \varphi^d \in g^{-1}Hg \text{ and } \forall 1 \le i < d : \varphi^i \notin g^{-1}Hg, \end{aligned}$$

$$\Leftrightarrow \ \exists g \in G: \ \psi(\varphi^d) \in \mathrm{Stab}_{S_n}(\psi(g^{-1})(1))$$

$$\text{and } \forall 1 \le i < d : \psi(\varphi^i) \notin \mathrm{Stab}_{S_n}(\psi(g^{-1})(1)),$$

$$\Leftrightarrow \ \exists j \in \{1,\dots,n\}: \ \psi(\varphi^d) \in \mathrm{Stab}_{S_n}(j) \text{ and } \forall 1 \le i < d : \psi(\varphi^i) \notin \mathrm{Stab}_{S_n}(j)$$

$$\Leftrightarrow \ \psi(\varphi) \text{ contains a } d\text{-cycle.}$$

This proves the proposition.

2.2Combinatorics in symmetric groups. Primes with a fixed residue degree *d* in a symmetric Galois group will be of importance to us in the future. These results lead us to think about elements in symmetric groups that have *d*-cycle, which we do now in the following portion of our analysis This section's content is also probably well-known. Due to the ease and simplicity of the procedures, I opted to include the proofs rather than spend time looking for relevant examples to use. Let d be a constant integer between one and two. Create an expression for,$d \geq 1$that is recursively defined for $i \geq 1$ and $1 \leq j \leq i$.

$$a(0) := 0, \quad b(i,j) := \frac{1}{j!d^j}(1 - a(i-j)), \quad a(i) := \sum_{k=1}^{i} b(i,k).$$

Lemma 2.2 With the preceding definitions we have

$$a(i) = \sum_{j=1}^{i} \frac{(-1)^{j+1}}{j!d^j} = 1 - \exp\left(\frac{-1}{d}\right) + \sum_{j=i+1}^{\infty} \frac{(-1)^j}{j!d^j}.$$

Proof. "This is a simple induction. For the convenience of the reader, we include the inductive step:

$$a(i+1)=\sum_{k=1}^{i+1} b(i+1,k)=\sum_{k=1}^{i+1}\frac{1}{k!d^k}(1-a(i+1-k))$$

$$=\sum_{k=1}^{i+1}\frac{1}{k!d^k}\left(1-\sum_{j=1}^{i+1-k}\frac{(-1)^{j+1}}{j!d^j}\right)=\sum_{k=1}^{i+1}\left(\frac{1}{k!d^k}-\sum_{j=1}^{i+1-k}\frac{(-1)^{j+1}}{k!j!d^{j+k}}\right)$$

$$=\sum_{m=1}^{i+1}\frac{1}{m!d^m}+\sum_{m=2}^{i+1}\frac{1}{m!d^m}\sum_{j=1}^{m-1}\binom{m}{j}(-1)^j=\sum_{m=1}^{i+1}\frac{(-1)^{m+1}}{m!d^m}.$$

For $i\to\infty$the convergence $a(i)\to 1-\exp(-1/d)$ is very quick because of the simple estimate of the error term $|\sum_{j=i+1} 1/j!\,dj| \leq$ **$2/(i+1)!d^{i+1}$** .We now relate the quantities $a(i)$ and $b(i, j)$ to proportions in the symmetric group. Let $n, j \in \mathbb{N}$. Define

$A_n(d) := \{g \in S_n \mid g$ contains at least one d-cycle$\}$,

$B_n(d, j) := \{g \in S_n \mid g$ contains precisely j d-cycles$\}$."

Lemma 2.3"*For all $n \geq 2d$ the following formulae hold, where $i := n/d\rfloor$:*

(a) $n! \cdot a(i) = \#A_n(d)$,

(b) $n! \cdot b(i, j) = \#B_n(d, j)$,

(c) $n! \cdot 2n-d-/n(n-1)\,(1-a(i-1)) = \#\{g \in B_n(d, 1) \mid$ *the unique d-cycle contains* 1 *or* $2\}$,

(d) $n! \cdot 1/n(n-1)\,(1-a(i-2)) = \#\{g \in B_n(d, 2) \mid$ *one d-cycle contains* 1, *the other* $2\}$."

Proof. "(a) and (b) are proved by induction for $n \geq 1$. For $n < d$ (i.e., $i = 0$), the equalities are trivially true. Now we describe the induction step:**$\#Bn(d,j)=1/j!\cdot(\binom{n}{d}).(d1)!\left(\binom{n-d}{d}\right).(d-1)!...X\left(\binom{n-(j-1)d}{d}\right).(d-1)!$ $X((n-jd))!.(1-a(i-j))!=\frac{n!}{j!dj(1-a(i-j))}$**."

"The first equality can be seen as follows: there are j! ways of ordering the j d-cycles.The number of choices for the first d-cycle is given by$\binom{n}{d}$***.(d-1)!****the one for the second is* $\left(\binom{n-d}{d}\right).(d-1)!$***...j d-cycles,*** *n − jd elements remain. Among these remaining elements we may only take those that do not contain any d-cycle; their number is* ***(n jd)!·(1 − a(i − j))*** *by induction hypothesis(c) The number of elements in the set in question is*

$$\left(2\binom{n-1}{d-1}-\binom{n-2}{d-2}\right)(d-1)!\cdot(n-d)!\cdot(1-a(i-1))=n!\frac{2n-d-1}{n(n-1)}(1-a(i-1))$$

because $\left(\binom{n-1}{d-1}\right).(d-1)!$ *is the number of choices for a d-cycle with one previously chosen element (i.e., 1 or 2) and*$\left(\binom{n-2}{d-2}\right).(d-1)!$*the number of choices for a d-cycle containing 1 and 2.*"

(d) "The number of elements in the set in question is

$$\binom{n-2}{d-1}(d-1)!\cdot\binom{n-2-(d-1)}{d-1}(d-1)!\cdot(n-2d)!\cdot(1-a(i-2))$$
$$= n!\frac{1}{n(n-1)}(1-a(i-2))$$

Because $\left(\binom{n-2}{d-2}\right).(d-1)!$*is the number of choices for a d-cycle containing 1 and not containing 2 and*$\left(\binom{n-2(d-1)}{d-1}\right).(d-1)!$*is the number of choices for a d-cycle containing 2 among the elements remaining after the first choice, and again (n−2d)! ·(1 − a(i − 2)) is the number of elements remaining after the two choices such that they do not contain any d-cycle.*

We write An ± (d) for the subsets of An(d) consisting of the elements having positive or negative signs."

Corollary 2.4"*Let d, n ∈N, n ≥ 2d ≥ 2 and puti := _ i := n/d⌋Then the estimates*

$$\left|\#\mathcal{A}_n^+(d)-\#\mathcal{A}_n^-(d)\right|\le n!\cdot\left(\frac{2n-d-1}{n(n-1)}(1-a(i-1))+\frac{1}{n(n-1)}(1-a(i-2))\right)$$
$$\le n!\cdot\frac{2}{n-1}$$

And

$$\left|\frac{\#\mathcal{A}_n^+(d)-\#\mathcal{A}_n^-(d)}{\#\mathcal{A}_n(d)}\right|\le\frac{1}{n-1}\cdot\frac{2}{1-\exp(-\frac{1}{d})-\frac{2}{(i+1)!d^{i+1}}}$$

Hold"

Proof. Consider the bijection $\phi : S_n \xrightarrow{g\mapsto g\circ(12)} S_n.$

"For j > 2 the image of A_n*+ (d) ∩*B_n*(d, j) under φ lands in* A_n*−n (d) because the multiplication with (1 2) can at most remove two d-cycles. Now consider g ∈*A_n*+(d)∩*B_n*(d, 2). Clearly φ(g) ∈*A_n*− (d) unlessone of the d-cycles contains 1 and the other one contains 2. For g ∈*A_n*+(d)∩Bn(d, 1)we find that φ(g) ∈*A_n*− (d) unless the single d-cycle of g contains 1 or 2. In view ofLemma 2.3, we thus obtain the inequality.*

$$\#\mathcal{A}_n^+(d)-\#\mathcal{A}_n^-(d)\le n!\cdot\left(\frac{2n-d-1}{n(n-1)}(1-a(i-1))+\frac{1}{n(n-1)}(1-a(i-2))\right)$$
$$\le n!\cdot\frac{2}{n-1}.$$

By exchanging the roles of + and − we obtain the first estimate. The second estimate then is an immediate consequence of Lemma 2.2 and the trivial estimate of the error term mentioned after it."

2.3. Density of primes with prescribed residue degree in composites of field extensions with symmetric Galois groups.

Lemma 2.5"*Let* $1 \leq d \in N$, *K be a field and L/K, F/K be two finite Galois extensions such that* $Gal(L/K) \cong Sn$ *with* $n \geq max(5, 2d)$ *and L is not a subfield of F.Let* $C \subseteq G := Gal(F/K)$ *be a subset and put*

$$c := \frac{\#C}{\#G} \quad and \quad a := \frac{\#A_n(d)}{\#S_n} = a(\lfloor \frac{n}{d} \rfloor).$$

Let X := Gal(LF/K) and Y be the subset of X consisting of those elements that project to an element in $A_n(d) \subseteq Sn \cong Gal(L/K)$ *or to an element in* $C \subseteq Gal(F/K)$*under the natural projections." Then*

$$\frac{\#Y}{\#X} = a + c - (1+\delta)ac,$$

where

$$\begin{cases} \delta = 0, & if\ L \cap F = K, \\ |\delta| \leq \frac{1}{n-1} \cdot \frac{2}{1-\exp(-\frac{1}{d}) - \frac{2}{(1+\lceil \frac{n}{d} \rceil)!\, d^{1+\lceil \frac{n}{d} \rceil}}}, & otherwise. \end{cases}$$

Proof. "The intersection L ∩ F is a Galois extension of K which is contained in L.The group structure of Sn (more precisely, the fact that the alternating group A_n *is the only non-trivial normal subgroup of Sn) hence implies that* $[L \cap F : K] \leq 2$*;for, if L ∩ F were equal to L, then L would be a subfield of F, which is excluded byassumption.*

Assume first L ∩ F = K, then Gal(LF/K) Gal(L/K)× Gal(F/K) and thus, #Y = #An(d)·#G+#Sn·#C−#An(d)·#C,from which the claimed formula follows by dividing by #X = #G·#Sn"

"Assume now that L∩F =: N is a quadratic extension of K. Then X is isomorphic to the index 2 subgroup of Gal(L/K)×Gal(F/K) consisting of those pairs of elements (g, h) such that g and h project to the same element in Gal(N/K). The elements of$A_n(d)$ *that project to the identity of Gal(N/K) are precisely those in* $A_n + (d)$*. In a similar way, we denote by C+ those elements of C projecting to the identity of Gal(N/K), and by C− the others. " Then we have*

$$\#Y = \#A_n(d) \cdot \frac{\#G}{2} + \frac{\#S_n}{2} \cdot \#C - \#A_n^+(d) \cdot \#C^+ - \#A_n^-(d) \cdot \#C^-.$$

Dividing by $\#X = \frac{\#S_n \cdot \#G}{2}$ we obtain

$$\frac{\#Y}{\#X} = a + c - (1+\delta)ac, \text{ where } \delta = \frac{\#C^+ - \#C^-}{\#C} \cdot \frac{\#A_n^+(d) - \#A_n^-(d)}{\#A_n(d)}.$$

The claim is now a consequence of Corollary 2.4.

Lemma 2.6"*Let* $(a_n)_{n \geq 1}$ *be a sequence of non-negative real numbers such that*$\sum_{n=1}^{\infty}$ *an diverges.*

(a) Let $\gamma > 0$, $b_0 \in R$*. Assume that* $a_n < 1/\gamma$ *for all* $n \geq 1$*. We define a sequence* $(b_n)n \geq 0$ *by the rule*

$$\boldsymbol{b_n := b_{n-1} + a_n - \gamma b_{n-1an}}$$

for, all $n \geq 1$*. Then the sequence* $(b_n)n \geq 1$ *tends to* $1/\gamma$ *for* $n \to \infty$*.*

(b) Let (δn)n≥1 be a sequence of real numbers tending to 0 and let $c_0 \in R$. Assume thatlim $\sup_{n\to\infty}$ an < 1. We define the (modified) inclusion–exclusion sequence as

$c_n := c_n-1 + a_n - (1 + \delta_n)c_n-1a_n$ for $n \geq 1$

Then the sequence $(c_n)_{n\geq 1}$ tends to 1.

Proof. (a) We let

$r_n := 1 - \gamma b_n = 1- \gamma(b_n-1 + a_n - \gamma b_n-1a_n) = (1 - \gamma b_n-1)(1 - \gamma a_n)$

$= (1 - \gamma b_0)(1 - \gamma_{a1})(1 - \gamma_{a2}) \cdots (1 - \gamma a_n)$."

To see that the limit of $(\gamma b_n)n\geq 0$ is 1, we take the logarithm of

"$(1-\gamma a_1)(1-\gamma a_2) \cdots (1 - \gamma a_n)$:

$$\sum_{i=1}^{n} \log(1-\gamma a_i) = -\gamma \sum_{i=1}^{n} a_i - \sum_{i=1}^{n}\sum_{j=2}^{\infty} \frac{(\gamma a_i)^j}{j} \leq -\gamma \sum_{i=1}^{n} a_i.$$

By our assumption this diverges to −∞ for n→∞, so that $\lim_{n\to\infty\, rn} = 0$, proving the lemma."

2.4. End of the proof.

"Let min $\left(1, \frac{1}{\limsup_{n\to\infty} a_n} - 1\right) > \epsilon > 0.$

There is N such that |δn| < ∈ and a_n< 1/1+∈ for all n ≥ N.By enlarging N if necessary we may also assume that $c_N \geq 0$. The reason for the latter is that $c_{N+}\sum_{n+i}^{n}$ an + 1if $c_{N+i} < 0$ for all0 ≤ i ≤ nWe consider the two sequences

$b_N := c_N$ and $b_n = b_{n-1} + a_n - (1 + \epsilon)b_{n-1} a_n$, for n > N.and

$d_N := c_N$ and $d_n = d_{n-1} + a_n - (1 - \epsilon)d_{n-1} a_n$, for n > N.

By (a) we know $\lim_{n\to\infty} b_n$ = 1/1+∈ and $\lim_{n\to\infty}$ dn = 1/1−∈ For n ≥ N by induction we obtain the estimate:

$0 \leq b_n \leq c_n \leq d_n$

Thus, there is M such that 1/1+∈−∈≤ cn ≤ 1/1−∈ + ∈ for all n ≥ M. As ∈ is arbitrary,we find $\lim_{n\to\infty\, cn} = 1$."

2.4. End of the proof

"Since dimℂ $S_k(1)$ tends to∞ for k→∞(for even k), Maeda'sconjecture implies the existence of new forms f_n of level one and increasing weight(automatically without complex multiplication because of level 1) such that their coefficient fields M_n := ℚf_n satisfy the assumptions of TheoremFor each n and each prime 𝔓 of $M_n$consider the Galois representation $\rho^{\mathrm{proj}}_{f_n,\mathfrak{P}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ *attached to f_n.implies that for each f_n and all but possibly finitely many P many 𝔓, its image is equal to PGL_2F(𝔓)if the residue field F 𝔓 of 𝔓 has odd degree over its prime field, and equal to PSL_2 (F 𝔓) if the residue degree is even. We will abbreviate this by PXL_2(F 𝔓)).*

Consequently, the set of primes (of ℚ)

$$\{p \mid \exists i \in \{1, \ldots, n\}, \exists \mathfrak{P}/p \text{ prime of } M_i \text{ s.t. } \rho^{\text{proj}}_{f_i, \mathfrak{P}} \cong \mathrm{PXL}_2(\mathbb{F}_{p^d})\}$$

has the same density as the corresponding set in Theorems, implying Theorem 1.1."

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